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ON ZONOIDS WHOSE POLARS ARE ZONOIDS

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ABSTRACT

Zonoids whose polars are zonoids cannot have proper faces other than vertices or facets. However, there exist non-smooth zonoids whose polars are zonoids. Examples in $\mathbb{R}^3$ and $\mathbb{R}^4$ are given.

Introduction

A **zonotope** in $\mathbb{R}^n$ is a vector sum of segments. A **zonoid** in $\mathbb{R}^n$ is a limit of zonotopes in $\mathbb{R}^n$ with respect to the Hausdorff metric. The sum of the centers of the segments defines a center of symmetry for a zonotope, and so by definition every zonoid is centrally symmetric, compact and convex. Consequently, every zonoid is the unit ball of some norm. The special structure of zonoids allows a more precise statement about what kind of norms enter the discussion of zonoids. Writing down the support function of a zonotope we see that zonoids are precisely unit balls of quotients of L_∞ spaces. Since every two-dimensional convex, centrally symmetric and compact set is a zonoid, every two-dimensional Banach space is isometric to a subspace of L_1. Therefore there is no zonoid theory in $\mathbb{R}^2$. For detailed discussions concerning zonoid theory see [1, 5, 12].

A well known theorem in Functional Analysis, due to Grothendieck, asserts that among infinite-dimensional Banach spaces, the ones which are isomorphic both to a subspace of L_1 and to a quotient-space of L_∞ are isomorphic to a Hilbert space. (For a proof see [8].) A natural question is whether this "isomorphic" theorem has an "isometric" analogue. E. Bolker conjectured in [1] (conjecture 6.8) that there is a finite-dimensional isometric version of Grothendieck's theorem.

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Bolker formulated his conjecture in the language of zonoids, which amounts to the conjecture that a zonoid whose dimension is at least 3 and whose polar is also a zonoid must be an ellipsoid.

Six years later, R. Schneider in [10] constructed examples of zonoids whose polars are also zonoids, that are not ellipsoids, and consequently Bolker's conjecture was disproved. The infinite-dimensional isometric problem is still open.

Schneider used spherical harmonics to prove that it is possible to apply smooth perturbations to the Euclidean ball in $\mathbb{R}^n$, such that the resulting bodies are zonoids whose polars are zonoids. Since the set of zonoids in $\mathbb{R}^n$, where $n \geq 3$, is closed and nowhere dense with respect to the Hausdorff metric, the perturbations had to be performed with respect to an essentially different metric. The metric employed by Schneider involved high-order derivatives of the support function. It seemed plausible after Schneider's work that zonoids whose polars are zonoids must be smooth, and hence strictly convex.

In §2 of this work an example of a non-smooth zonoid whose polar is also a zonoid is presented. It consists of forming the Minkowski sum of a three-dimensional Euclidean unit ball and a concentric circle of radius 1. Similar examples exist in $\mathbb{R}^4$.

Although smoothness of zonoids whose polars are zonoids cannot be guaranteed, it nevertheless cannot be lost in an arbitrary fashion. §1 contains some information in this direction. The result is

THEOREM: *Suppose $K = B + C$ is a convex body in $\mathbb{R}^n$, where $n \geq 3$ and B, C are convex, compact and centrally symmetric subsets of $\mathbb{R}^n$. If $1 \leq \dim C \leq n-2$ then the polar of K is not a zonoid.*

It is well known that every face of a zonoid Z is a translate of a zonoid which is a summand of Z ([1], Th. 3.2, and also [11], p. 189). Therefore, the theorem implies:

COROLLARY: *If $n \geq 3$ and Z is an n-dimensional zonoid whose polar is also a zonoid, then the boundary of Z does not contain proper faces whose dimension is different from $n-1$ or zero.*

As far as the dimension is concerned, the examples of §2 show that these statements cannot be improved.

Another immediate corollary of the theorem is that the polar of a zonotope whose dimension is at least 3 is not a zonotope. This had been proved long ago by M. A. Perles, and by E. Bolker (cf. [1]). Both proofs are based on the special polytopal structure of a zonotope. In particular, Perles shows that every zonotope whose dimension is at least 3 has more vertices than facets.

§1. Proof of the theorem

A fundamental property of zonoids which is exploited below appears as Theorem 3.2 in [1]. It is stated here as a lemma.

LEMMA 1.1: *Every proper face of a zonoid K is a translate of a zonoid of lower dimension which is a summand of K.*

A summand can be either direct or not. B is said to be a direct summand of K if $K = B + C$ and $\dim B + \dim C = \dim K$. In such cases the Minkowski sum is written in the form $K = B \oplus C$. Due to this distinction between types of summands, the proof of the theorem will be divided into two parts. The first part consists of a proposition which settles the case of direct summands. The restriction $\dim C \leq n-2$ that appears in the formulation of the theorem does not play any role in the setting of direct summands.

PROPOSITION 1.2: *A polar of a zonoid whose dimension is at least 3 does not have non-trivial direct summands.*

Proof: Assume that K is a polar of a zonoid, $\dim K \geq 3$ and K does have non–trivial direct summands. Applying a suitable linear transformation, it can be assumed that $K = B \oplus C$ where $\operatorname{span} B$ and $\operatorname{span} C$ are mutually orthogonal. Let P denote the orthogonal projection onto $\operatorname{span} B$ and let $Q = I - P$. Then the norm of K can be written as

$$\|x\|_K = \max\{\|Px\|_B, \|Qx\|_C\}, \quad \forall x \in \mathbb{R}^n. \tag{1.1}$$

Choose any point x on the boundary of B, and any point y on the boundary of C. Then

$$\|(x+y)+(x-y)\|_K+\|(x+y)-(x-y)\|_K = 4 = 2\left(\|x+y\|_K + \|x-y\|_K\right). \tag{1.2}$$

By assumption, K° is a zonoid. Hence there is a positive measure ν on the sphere such that

$$\|z\|_K = \int_{S^{n-1}} |\langle z, u\rangle| \, d\nu(u), \quad \forall z \in \mathbb{R}^n. \tag{1.3}$$

For every $x \in \partial B$ and $y \in \partial C$ consider two functions defined on the sphere by

$$f_{y,x}(u) = \langle x+y, u\rangle \quad \text{and} \quad g_{y,x}(u) = \langle x-y, u\rangle.$$

Then by (1.3) and (1.2), for the norm in $L^1(d\nu)$, and for $f = f_{x,y}$ and $g = g_{x,y}$, one has

$$\|f+g\| + \|f-g\| = 2\left(\|f\| + \|g\|\right). \tag{1.4}$$

Such an equality can occur only if

$$f(u)g(u) = 0 \quad \text{for } \nu\text{-almost every } u \in S^{n-1}. \tag{1.5}$$

This implies

$$\operatorname{supp}\nu \subset \{u \in S^{n-1} : |\langle u,x\rangle| = |\langle u,y\rangle|\} \quad \forall y \in \partial C, \quad \forall x \in \partial B. \tag{1.6}$$

Observe that if $x_1, \ldots, x_n$ is a linear basis for $\mathbb{R}^n$ then the set

$$\{u \in S^{n-1} : |\langle u,x_1\rangle| = |\langle u,x_2\rangle| = \cdots = |\langle u,x_n\rangle|\} \tag{1.7}$$

is finite (it contains at most 2^n points). Since the dimension of K is at least three, the dimension of one of the summands must be at least two, and so (1.6) cannot hold for every choice of points $x \in \partial B$ and $y \in \partial C$, due to the previous observation. This contradiction proves the proposition. ∎

The second part of the proof consists of proving the theorem for non-direct summands. Our original proof was for a one-dimensional summand, i.e, a segment, but as was kindly pointed out by R. Schneider, the same argument yields the result as stated here.

Let K be a centrally symmetric, compact and convex subset of $\mathbb{R}^n$. Given any subset $U \subset \mathbb{R}^n$, consider the following subset of the boundary of K:

$$A(K,U) = \{x \in \partial K : K \text{ has an outer normal at } x \text{ which is orthogonal to } \operatorname{span} U\}.$$

Concerning such sets the following Lemma holds.

LEMMA 1.3: *Suppose K is a centrally symmetric, compact and convex subset of $\mathbb{R}^n$, such that $\dim K \geq 2$. Let $U \subset \mathbb{R}^n$ denote a subset whose span is of dimension at most $n-2$. Then for every $(n-1)$-dimensional subspace H the intersection $A(K,U) \cap H$ is not empty.*

Proof: One may assume that U is a subspace of dimension $n-2$. Let H be a subspace of dimension $n-1$.

First assume that K is smooth and strictly convex. Let G_K denote the Gauss map, which takes every point x on ∂K to the outward unit normal of K at x. Then G_K is a homeomorphism between ∂K and S^{n-1}.

The intersection $U^\perp \cap S^{n-1}$ is a great circle. Since the set $A(K,U)$ is the inverse image of this intersection under the Gauss map, it is a connected, centrally symmetric subset of the boundary of K and hence meets H. This proves the assertion under the special assumption on K.

The proof for general K is now concluded by an approximation argument. Let K be as in the formulation of the lemma. Choose a sequence $(K_i)_{i\in\mathbb{N}}$ of centrally symmetric, smooth and strictly convex bodies converging to K. For each $i \in \mathbb{N}$ there exists a pair $(x_i, u_i) \in H \times U^\perp$ such that $x_i \in \partial K_i$ and u_i is an outer normal vector K_i at x_i, thus $h(K_i, u_i) = \langle x_i, u_i\rangle$, where $h(K_i, \cdot)$ denote the support functions. The sequence (x_i, u_i) has a convergent subsequence, and one may assume that this sequence itself converges to a pair (x, u). Since the support function h is simultaneously continuous in both variables, one gets $h(K,u) = \langle x, u\rangle$. If $h(K,u) \neq 0$, then $x \in \partial K$, and from $(x,u) \in H \times U^\perp$ one gets $x \in A(K,U) \cap H$. On the other hand, if $h(K,u) = 0$, then since $\dim K \geq 2$, there exists a point $y \in \partial K \cap H$, and this point is in the set $A(K,U) \cap H$. ∎

Proof of the theorem: The main idea is similar to the one which appeared above, in the proof of the proposition. Suppose K° (the polar of K) is a zonoid, K is n-dimensional and has a non-direct summand C where $1 \leq \dim C \leq n-2$. Then $K = B + C$, for some centrally symmetric compact and convex subset B of $\mathbb{R}^n$. For every $x \in A(B,C)$, the set $x + C$ lies entirely on the boundary of K. Therefore,

$$\|(x+c)+(x-c)\|_K = 2 = \|x+c\|_K + \|x-c\|_K, \quad \forall x \in A(B,C), \quad \forall c \in C. \tag{1.8}$$

Arguing similarly as in the proof of the proposition, an equality in the triangle inequality in $L_1(S^{n-1}, \nu)$ is obtained, where the vectors are the functions

$$f_{x,c}(u) = \langle x + c, u\rangle \quad \text{and} \quad g_{x,c}(u) = \langle x - c, u\rangle, \tag{1.9}$$

and $c \in A(B,C)$ is arbitrary. Since K° is a zonoid, there exists a positive Borel measure ν on the sphere which satisfies an equation of the form (1.3). Being positive, the measure must assign all its mass to the set of points where the functions from (1.9) have the same sign. Therefore,

$$\operatorname{supp} \nu \subset \{u \in S^{n-1} : |\langle u, x\rangle| \geq |\langle u, c\rangle|\} \quad \forall x \in A(B,C), \quad \forall c \in C. \tag{1.10}$$

By Lemma 1.3, for every $u \in S^{n-1}$ there exists a point $x \in A(B,C)$ such that $x \perp u$, and so from (1.10) the measure is seen to be concentrated on the section $C^{\perp} \cap S^{n-1}$, whose dimension is $(n - \dim C)$. But this contradicts the fact that the dimension of K is n. This completes the proof of the theorem. ∎

§2. The barrel zonoid

The purpose of this section is to present an example of a non-smooth zonoid whose polar is also a zonoid. First, the so-called "barrel zonoid" is introduced and some of its properties are discussed. Afterwards the analytic tools to be used are presented, followed by a calculation which yields the desired example. The main tool here is an inversion formula for the cosine transform, due to Goodey and Weil [4].

Let B_2^n denote the n-dimensional euclidean unit ball. For a positive number $r > 0$, consider the zonoid $\mathcal{B}_{n,r} = B_2^n + rB_2^{n-1}$. It is invariant under rotations which keep the n'th coordinate fixed. Such bodies are called **rotationally symmetric**. If $0 \leq \varphi \leq \pi$ denotes the vertical angle in spherical coordinates, then the restriction of the norm of $\mathcal{B}_{n,r}$ to the unit sphere depends only on φ. Therefore it can be identified with a function f_r defined on the interval $[0, \pi]$, and by symmetry, attention can be restricted to the interval $[0, \pi/2]$. The rotational symmetry implies that for all dimensions $n \geq 3$, the norm of $\mathcal{B}_{n,r}$ is represented by the same function f_r. A simple two-dimensional calculation shows that

$$(2.1) \qquad f_r(\varphi) = \begin{cases} \cos\varphi, & \text{if } 0 \leq \varphi \leq \tan^{-1} r, \\ \dfrac{1}{r\sin\varphi + \sqrt{1 - r^2\cos^2\varphi}}, & \text{if } \tan^{-1} r \leq \varphi \leq \pi/2. \end{cases}$$

In case $n = 3$ and $r = 1$ the resulting body is barrel-shaped. Henceforth the name "barrel zonoid" will refer to a body of the form $\mathcal{B}_{n,r}$, and $\mathcal{B}_{n,1}$ will be denoted by $\mathcal{B}_n$.

The support function of the barrel-zonoid is the sum of the support functions of its summands. Therefore its restriction to the sphere is the function $1 + r\sqrt{1 - u_n^2}$, where u_n denotes the nth coordinate. It is not differentiable at the points $\pm e_n$, which geometrically means that there is no unique supporting hyperplane to the polar at the points $\pm e_n$. This of course corresponds to the fact that $\mathcal{B}_{n,r}$ itself is not strictly convex. The polar $\mathcal{B}_{n,r}^{\circ}$ is also rotationally symmetric, and in case $n = 3$, $r = 1$, when it is intersected by a plane parallel to $(0,0,1)$ and passing through the origin, the result is a symmetric, parabolic curve whose equation is easily obtained by means of the radial function of the polar, and is given by

$|y| = (1 - x^2)/2$, for $|x| \leq 1$. Rotating this curve about the interval $|x| \leq 1$ yields the polar of the three-dimensional barrel (of radius $r = 1$), and so an explicit figure of the polar may be obtained. Its shape resembles that of an American football.

Let $\|\cdot\|$ denote a norm in $\mathbb{R}^n$. In order to prove that the polar of the unit ball determined by the given norm is a zoonoid, one needs to find a positive, symmetric measure μ on the sphere such that

$$\|u\| = \int_{S^{n-1}} |\langle u, v\rangle| \, d\mu(v), \quad \forall u \in \mathbb{R}^n. \tag{2.2}$$

The r.h.s of this equation is the **cosine transform** of the measure μ. Usually the cosine transform $T : C_e^\infty(S^{n-1}) \to C_e^\infty(S^{n-1})$ is defined on the space of infinitely differentiable even functions on the unit sphere S^{n-1} by

$$(Tf)(u) = \int_{S^{n-1}} |\langle u, v\rangle| f(v) \, d\lambda_{n-1}(v),$$

where λ_{n-1} is the spherical Lebesgue measure on S^{n-1}. It is clear that the same formula can be used to transform more general objects than C^∞ functions on the sphere, such as measures. Therefore the equation (2.2) can be viewed as

$$T\mu = \|\cdot\|. \tag{2.3}$$

This equation is well known and has been the subject of many investigations. In 1937, A.D. Alexandrov proved that there is at most one symmetric measure which solves (2.3). Since then, several other proofs of the same fact were found. See, e.g., [7].

There does not always exist a symmetric measure μ that solves (2.3) for a given norm. In fact, it is known that if the norm is of a polytope, then a solution exists only if this polytope is a polar of a zonotope ([11], corollary 3.5.6, p. 188). However, Weil [13] showed that for every norm there exists a symmetric distribution ρ (i.e., a continuous linear functional on the space $C_e^\infty(S^{n-1})$) whose domain can be extended to include the functions $|\langle u, \cdot\rangle|, u \in S^{n-1}$, such that $\rho(|\langle u, \cdot\rangle|) = \|u\|$. The fact that every distribution can be viewed as the cosine transform of a distribution follows from the self-duality of $T : C_e^\infty(S^{n-1}) \to C_e^\infty(S^{n-1})$ (a simple consequence of Fubini's theorem) and a result by Schneider, asserting that T is onto $C_e^\infty(S^{n-1})$. Therefore (2.3) can always be solved with a distribution instead of a measure, for any given norm, and the symbol $T^{-1}(h_K)$ acquires a precise meaning for every given support function of a centrally symmetric convex body K. The distribution $T^{-1}(h_K)$ is called the **generating distribution** of the convex body K. It is well known that positive distributions are in fact positive measures. Therefore in the context of zonoids Weil's result is particularly useful, because it provides a priori a functional whose positiveness is to be checked. For an illustration of this technique, see [4], Th. 5.1.

The problem is now to prove that for the norm of the three-dimensional barrel zonoid, the generating distribution $T^{-1}(\|\cdot\|)$ is positive. To this end an inversion formula for the cosine transform shall be used, which involves the **spherical Radon transform** $R: C_e^\infty(S^{n-1}) \to C_e^\infty(S^{n-1})$, defined by

$$(Rf)(u) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}\cap u^\perp} f(v)\, d\lambda_{n-2}(v), \quad u \in S^{n-1},$$

where ω_{n-1} is the total spherical Lebesgue measure of the unit sphere in $\mathbb{R}^{n-1}$. Let Δ_n denote the spherical Laplace operator on S^{n-1}. In [4], Goodey and Weil prove the following inversion formula:

$$T^{-1} = \frac{1}{2\omega_{n-1}}(\Delta_n + n - 1)R^{-1}. \tag{2.4}$$

It is well known that the Radon transform is a self-adjoint continuous bijection of $C_e^\infty(S^{n-1})$ to itself (see [6]). Since T and Δ_n also have this property, the inversion formula can be applied to the dual space of its natural domain, that is, to the space of even distributions. In particular it can be applied to any given norm, restricted to the sphere.

As for the inversion of the Radon transform, it is explained by Gardner in [2] that if f is a rotationally symmetric function on S^{n-1} and $f = Rg$, then g is also rotationally symmetric and the equation $f = Rg$ becomes

$$f(\sin^{-1} x) = \frac{2\omega_{n-2}\omega_{n-1}^{-1}}{x^{n-3}} \int_0^x g(\cos^{-1} t)(x^2 - t^2)^{(n-4)/2}\, dt, \tag{2.5}$$

for $0 < x \le 1$ and $g(\pi/2) = f(0)$. There is an inversion formula for this equation (see [2]). However, for $n = 4$ it is trivial to invert the equation (2.5) because if $xf(\sin^{-1} x)$ is differentiable, then (2.5) immediately implies

$$g(\cos^{-1} x) = \frac{d}{dx}(xf(\sin^{-1} x)), \tag{2.6}$$

for $0 \le x \le 1$. In proving the next claim, this formula will be used.

CLAIM: *The barrel zonoid $\mathcal{B}_{4,r}$ is a polar of a zonoid if and only if $r \leq 1$.*

Proof: (2.6) shall now be applied for $n = 4$ and for $f = f_r$, the norm of $\mathcal{B}_{4,r}$, given by (2.1). Let $x_r = r/\sqrt{1+r^2}$. Then

$$f_r(\sin^{-1} x) = \begin{cases} \sqrt{1-x^2}, & \text{if } 0 \leq x \leq x_r, \\ \dfrac{1}{rx + \sqrt{1 - r^2 + r^2x^2}}, & \text{if } x_r \leq x \leq 1. \end{cases}$$

The function $f_r(\sin^{-1} x)$ has a continuous derivative in $[0,1]$. Hence, by (2.6),

$$g(\cos^{-1} x) = \begin{cases} \dfrac{1-2x^2}{\sqrt{1-x^2}}, & \text{if } 0 \leq x \leq x_r, \\ \dfrac{1-r^2}{A(x,r)(rx + A(x,r))^2}, & \text{if } x_r \leq x \leq 1, \end{cases}$$

where $A(x,r) = \sqrt{1-r^2+r^2x^2}$.

In cylindrical coordinates, $u = (\sqrt{1-x^2}\xi, x)$, where $\xi \in S^2$; the four-dimensional spherical Laplacian is given by

$$\Delta_4 = (1-x^2)\frac{\partial^2}{\partial x^2} - 3x\frac{\partial}{\partial x} + \frac{1}{1-x^2}\Delta_3.$$

In this formula Δ_3 is applied to coordinates of ξ; these are independent of x. Hence, when applying the Laplacian to a rotationally symmetric function, the term containing Δ_3 disappears. Therefore the differential operator which is to be applied to $g(\cos^{-1} x)$ is given by

$$D = \frac{1}{8\pi}\left((1-x^2)\frac{d^2}{dx^2} - 3x\frac{d}{dx} + 3\right).$$

The calculation of the corresponding derivatives of $G(x) = g(\cos^{-1} x)$ has to be done in the distribution sense. The first derivative is an absolutely continuous measure whose density is given by

$$\frac{dG}{dx} = \begin{cases} \dfrac{2x^3-3x}{(1-x^2)^{3/2}}, & \text{if } 0 \leq x < x_r, \\ \dfrac{r(2A(x,r)+rx)(rx - A(x,r))}{A^3(x,r)(A(x,r)+rx)}, & \text{if } x_r < x \leq 1. \end{cases}$$

Due to the jump at the point $x = x_r$, the second derivative is a sum of a continuous measure and a measure concentrated at the point $x = x_r$ (see [3], §2). Therefore,

$$\frac{d^2G}{dx^2} = \begin{cases} \dfrac{3}{(1-x^2)^{5/2}} & \text{if } 0 \leq x < x_r \\ \dfrac{3r^2(1-r^2)}{A^5(x,r)}, & \text{if } x_r < x \leq 1 \end{cases} \quad + \quad c(r)\,\delta_{x_r}.$$

The constant $c(r)$ is given by

$$c(r) = \lim_{x\to x_r^+} \frac{dG}{dx} - \lim_{x\to x_r^-} \frac{dG}{dx} = r(r^2+1)^2.$$

Having all the derivatives the differential operator D can be applied directly. The result is

$$(2.7)\quad T^{-1}(f_r) = D(G) = \begin{cases} 0 & \text{if } 0 \le x < x_r \\ \dfrac{3(1-r^2)}{8\pi A^5(x,r)} & \text{if } x_r < x \le 1 \end{cases} + \frac{r(r^2+1)}{8\pi}\delta_{x_r}.$$

Evidently, the generating distribution is a positive measure if and only if $r \le 1$. The proof of the claim is complete. ∎

Remarks: (1) In the special case where $n = 4, r = 1$, the measure which is obtained in (2.7) is particularly simple. It is therefore easy to check directly that it represents the polar of $\mathcal{B}_4$. Here is the calculation.

The formula (2.7) shows that the measure in question is concentrated on the set $\{v \in S^3 : |v_4| = \sqrt{1/2}\}$. By rotational symmetry, its restriction to each one of the three-dimensional spheres comprising its support is proportional to the corresponding Lebesgue spherical measure. In order to check this, let $e_4 = (0,0,0,1)$ and consider the integral

$$(2.8)\qquad \int_{S^3\cap e_4^\perp} |\langle y+e_4, u\rangle|\, d\lambda_2(y).$$

Using invariance the point u can be replaced by the point $(0,0,\sqrt{1-t^2},t)$, where $t = u_4$. Applying spherical coordinates to (2.8) yields the following integral,

$$2\pi \int_0^\pi |\sqrt{1-t^2}\cos\psi + t|\sin\psi\, d\psi = 2\pi \begin{cases} 2|t| & \text{if } |t| \ge 1/\sqrt{2} \\ \frac{1}{\sqrt{1-t^2}} & \text{if } |t| \le 1/\sqrt{2} \end{cases}$$
$$= 4\pi f_1(\cos^{-1}|t|),$$

where f_1 is the norm of the barrel given by (2.1). Hence a (positive) multiple of the spherical Lebesgue measure on each one of the spheres $\{u \in S^3 : u_4 = \pm\sqrt{1/2}\}$ represents the polar $\mathcal{B}_4^\circ$, as was required to check.

(2) Since $\mathcal{B}_{n,r}$ is a central section of $\mathcal{B}_{n+1,r}$ on which there exists an orthogonal projection, the fact that $\mathcal{B}_{n+1,r}^\circ$ is a zonoid implies the same for $\mathcal{B}_{n,r}^\circ$. Consequently, $\mathcal{B}_{3,r}^\circ$ is a zonoid for $r \le 1$. For the special case $r = 1$ it is possible to apply

the same reasoning as above and obtain an explicit formula for the generating distribution of $\mathcal{B}_3^\circ$. Its density is given by

$$T^{-1}f(\varphi) = \begin{cases} c\dfrac{\cos^2\varphi + \sqrt{\cos 2\varphi}}{(1+\sqrt{\cos 2\varphi})^2\cos^3\varphi\sqrt{\cos 2\varphi}}, & \text{if } 0 \le \varphi < \frac{\pi}{4}, \\ 0, & \text{if } \frac{\pi}{4} \le \varphi \le \frac{\pi}{2}. \end{cases}$$

Here $c > 0$ is a positive constant. Hence the generating distribution of $\mathcal{B}_3^\circ$ has an L_1 (but not L_2) density.

(3) For $n \ge 6$ and $r > 0$, the polar of $\mathcal{B}_{n,r}$ is not a zonoid. Indeed, the calculation of this section for the case $n = 6$ results in a generating distribution that involves a derivative of a measure concentrated at a point. Consequently, the generating distribution of $\mathcal{B}_{6,r}^\circ$ is not a measure. A generalization of this calculation to higher dimensions can be used to answer a question raised by Goodey and Weil. For more details see [9].

It is plausible that non-smooth zonoids whose polars are zonoids exist in every dimension. However, the author is unaware of any examples other than the barrel zonoids.

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